The modal logic of planar polygons

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Introduction

We study the modal logic of the closure algebra $\mathbf{P_2}$, generated by the set of all polygons of the Euclidean plane $\mathbb{R}^2$. We show that:

- The logic is finitely axiomatizable
- It is complete with respect to the class of all finite "crown" frames we define
- It does not have the Craig interpolation property
- Its validity problem is PSpace-complete

Preliminaries

Modal Language	p,q,... $\wedge$ $\vee$ $\neg$ $\Diamond$ $\Box$	$\Diamond\Diamond p \rightarrow \Diamond p$
	⇩	⇩
Topological space X	P, Q,... $\cap$ $\cup$ $\setminus$ $\mathbb{C}$ $\mathbb{I}$	$\mathbb{C}\mathbb{C}A \subseteq \mathbb{C}A$

As is well known, logic **S4** is characterized by reflexive-transitive Kripke frames.

Preliminaries

The modal logic of the class of all topological spaces is **S4**.
Moreover, for any Euclidean space $\mathbb{R}^n$, we have Log($\mathbb{R}^n$) = **S4**.

[McKinsey and Tarski in 1944]

We study the topological semantics, according to which modal formulas denote regions in a topological space.

$(\mathcal{P}(\mathbb{R}^2), \mathbb{c}) \rightarrow (A, \mathbb{c})$

General spaces

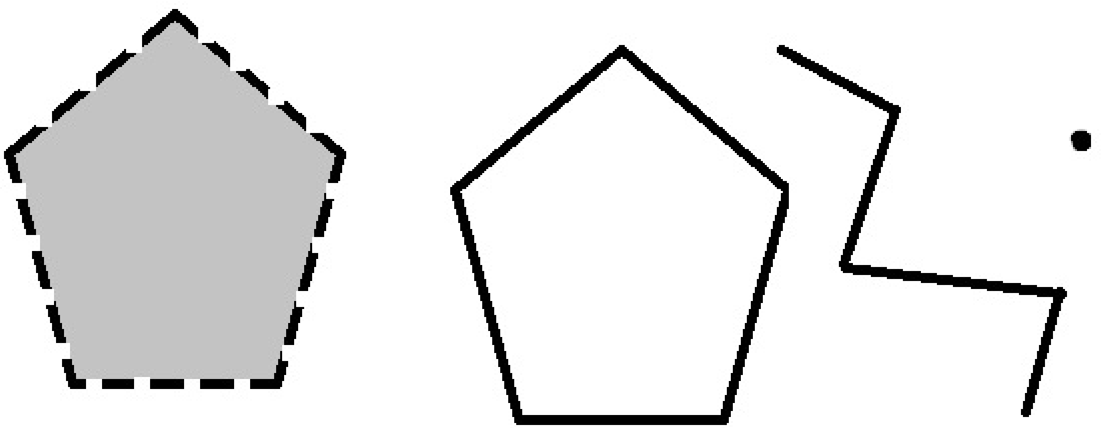
Topological spaces together with a fixed collection of subsets that is closed under set-theoretic operations as well as under the topological closure operator.

General models

Valuations are restricted to modal subalgebras of the powerset.

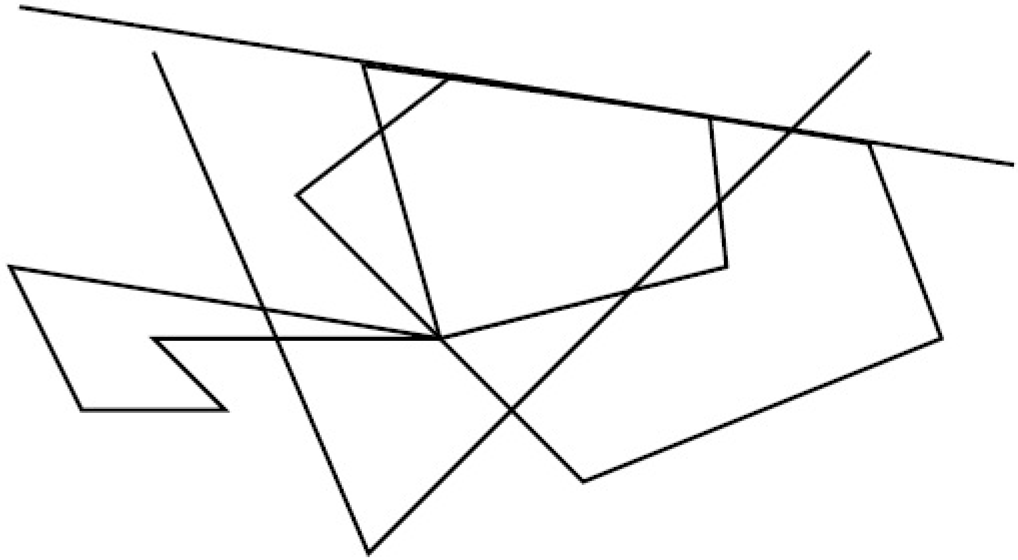
Preliminaries

Lets generate a closure algebra by polygons of $\mathbb{R}^2$ and denote it $\mathbf{P_2}$.



The 2-dimensional polytopal modal logic $\mathbf{PL_2}$ is defined to be the set of all modal formulas which are valid on $(\mathbb{R}^2, \mathbf{P_2})$.

Preliminaries



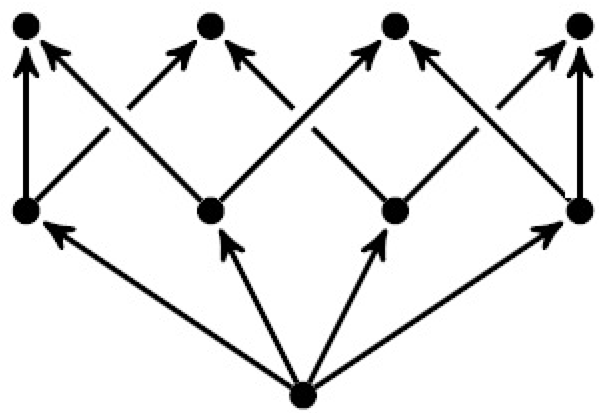
What is the modal logic of the polygonal plane?

R. Kontchakov, I. Pratt-Hartmann and M. Zakharyaschev, Interpreting Topological Logics Over Euclidean Spaces., in: Proceeding of KR, 2010

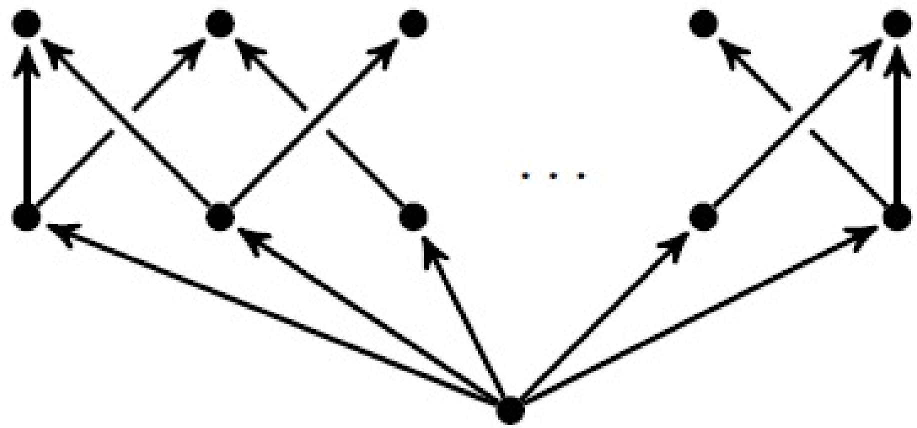
J. van Benthem, M. Gehrke and G. Bezhanishvili, Euclidean Hierarchy in Modal Logic, Studia Logica (2003), pp. 327-345

Preliminaries

The logic of chequered subsets of $\mathbb{R}^2$



Crown frames



$\mathcal{G}_n$

Let Λ be the logic of all "crown" frames.

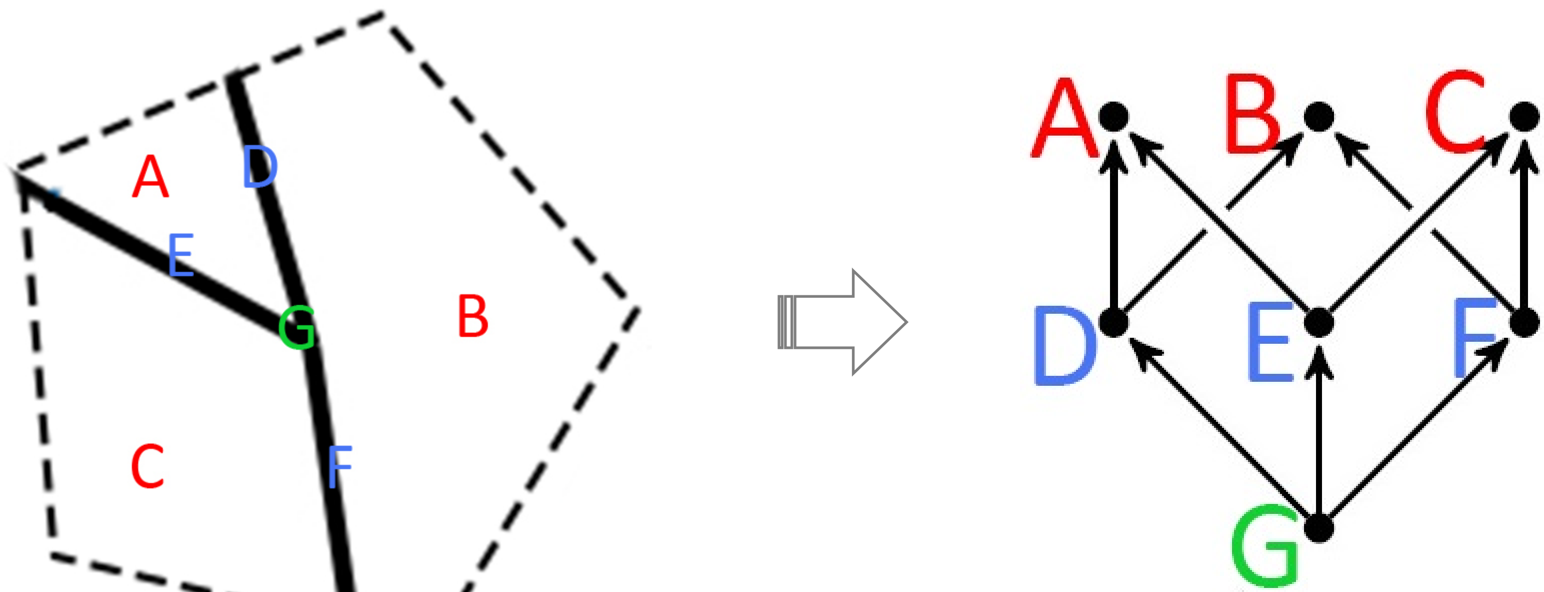
Theorem: Λ coincides with $\mathbf{PL_2}$.

Preliminaries

The map $f : X_1 \to X_2$ between topological spaces $X_1 = (X_1, \tau_1)$ and $X_2 = (X_2, \tau_2)$ is said to be an interior map, if it is both open and continuous.

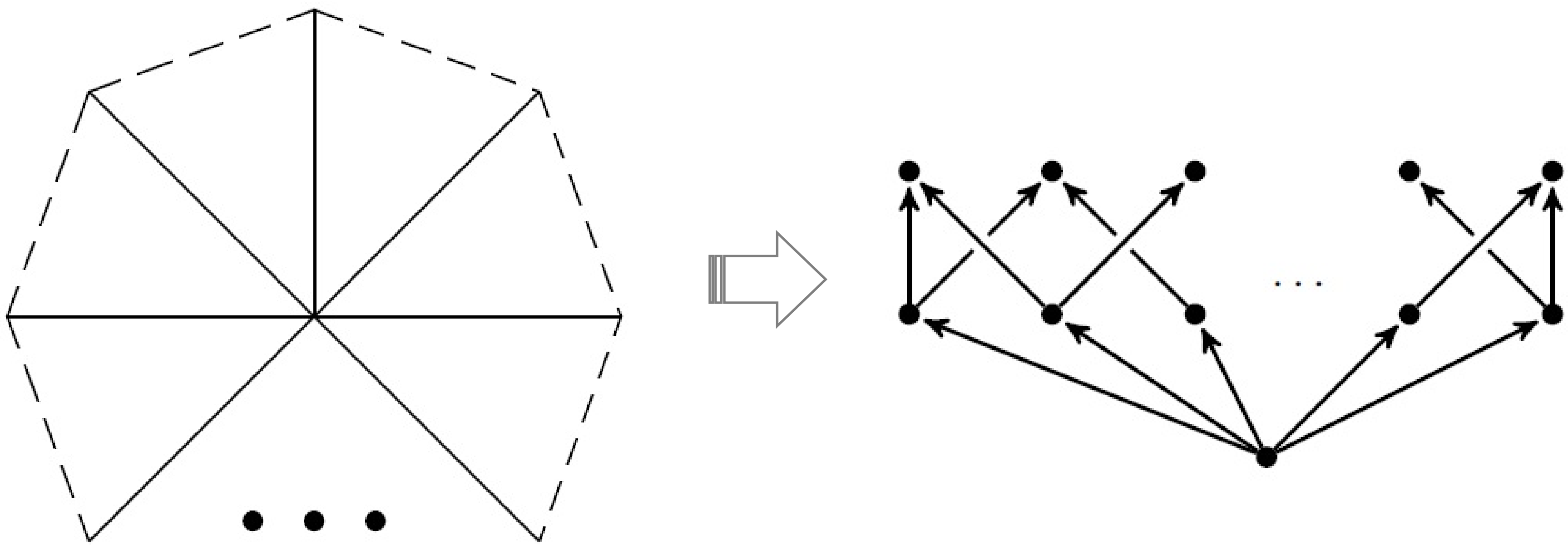
- Let X and Y be topological spaces and let $f : X \twoheadrightarrow Y$ be an onto partial interior map.
- Then for an arbitrary modal formula φ we have $Y \vDash \varphi$ whenever $X \vDash \varphi$.
- It follows that $\mathrm{Log}(X) \subseteq \mathrm{Log}(Y)$.

Example



The main results

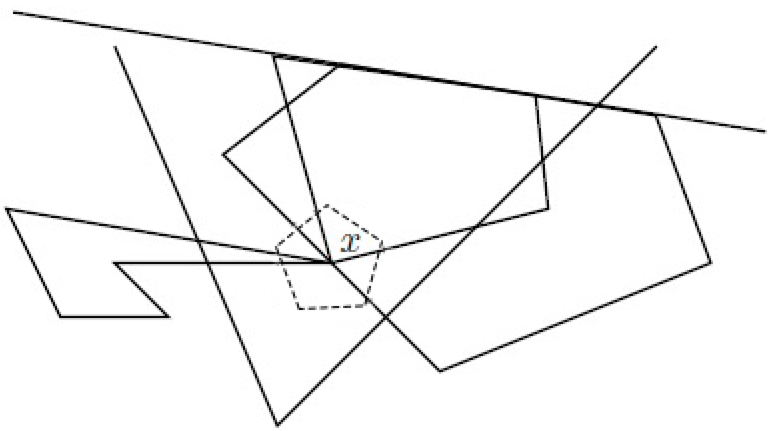
Theorem: Any crown frame is a partial interior image of the polygonal plane.

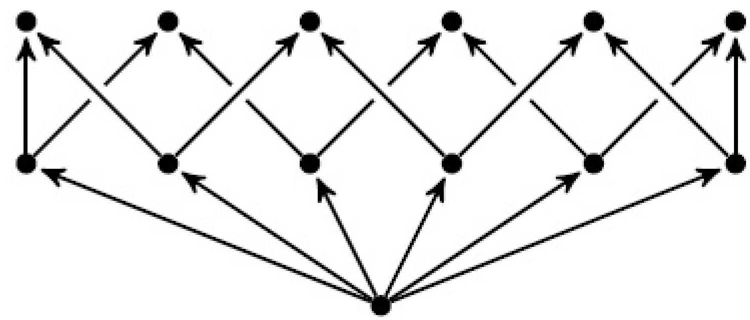


Corollary: $\mathbf{PL_2} \subseteq \Lambda$.

The main results

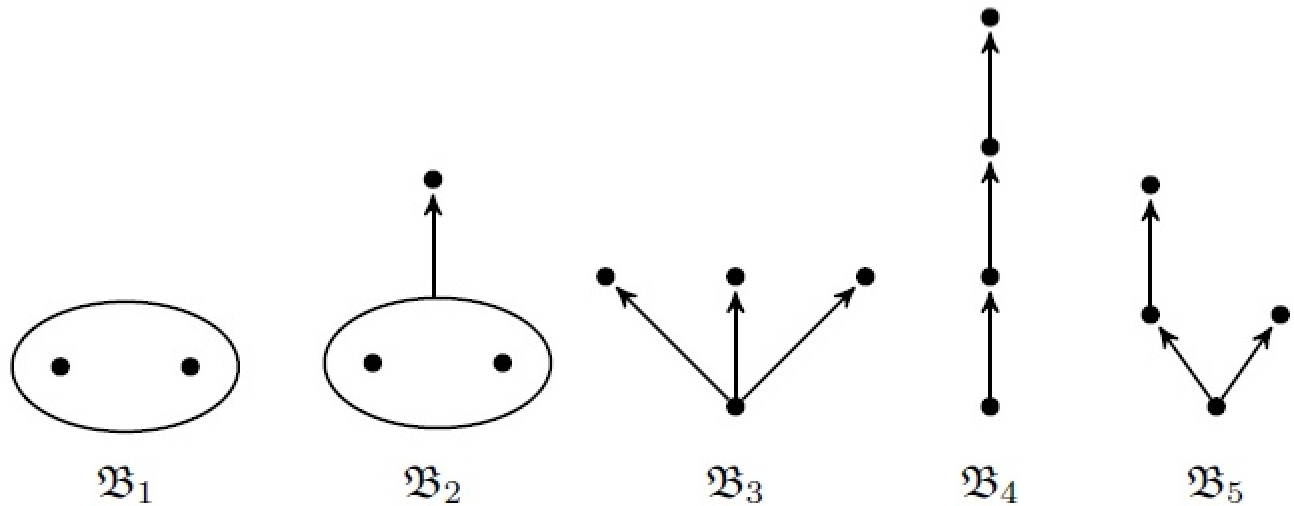
Theorem: Let φ be satisfiable on a polygonal plane. Then φ is satisfiable on one of the crown frames.



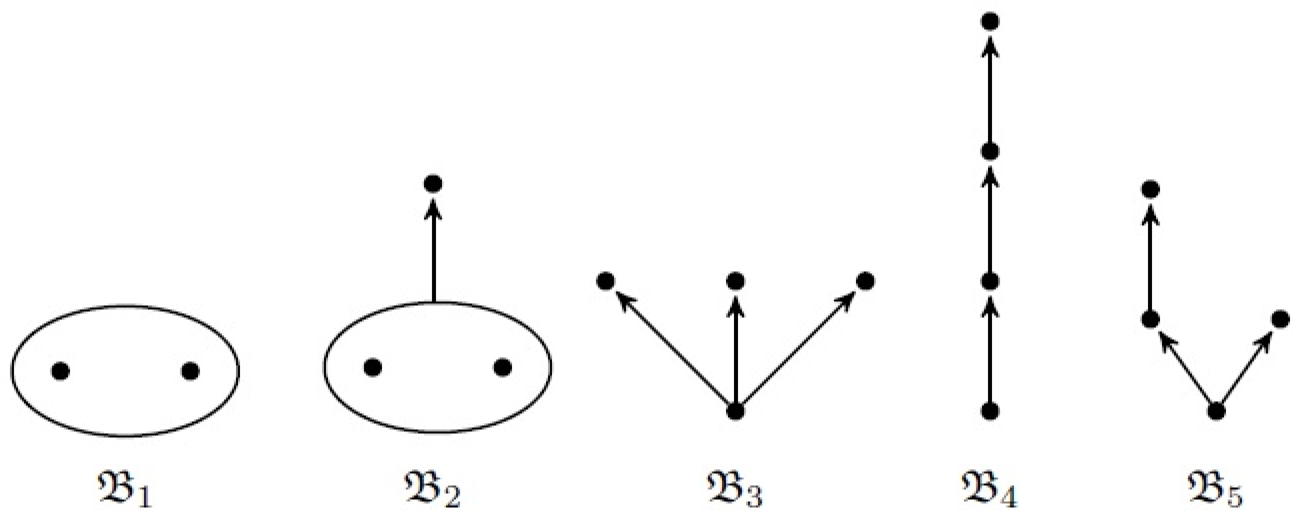


Corollary: $\Lambda \subseteq$ **$\mathbf{PL_2}$.** Thus, the logic of the polygonal plane is determined by the class of finite crown frames. Hence this logic has FMP.

Forbidden frames



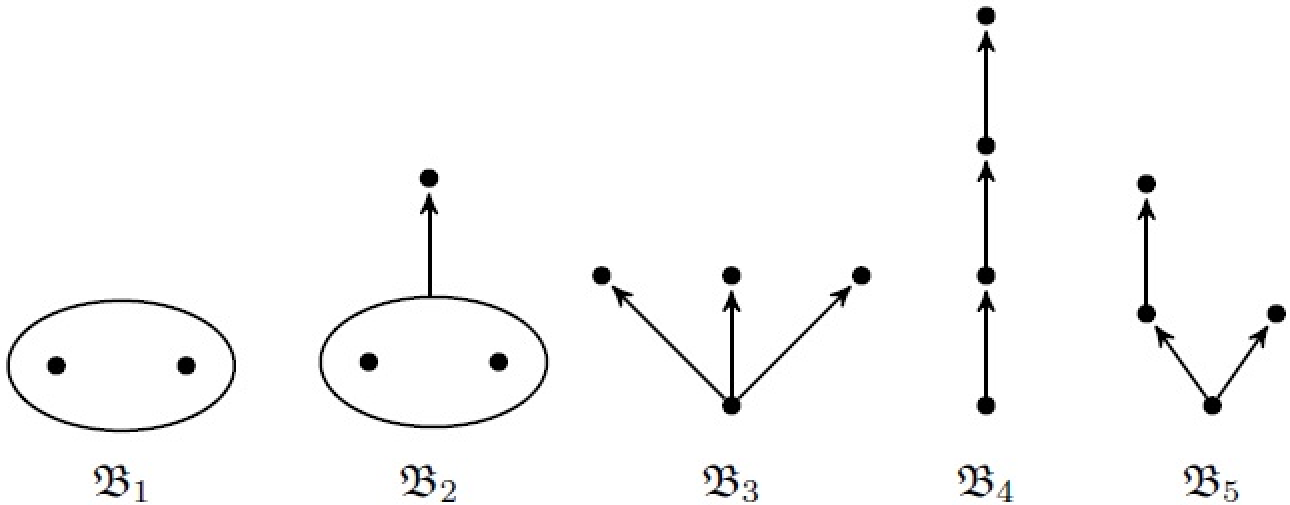
Axiomatization



We claim that the logic axiomatized by the Jankov-Fine axioms of these five frames coincides with PL_2.

$$\xi = \neg\xi(\mathfrak{B}_1) \wedge \neg\xi(\mathfrak{B}_2) \wedge \neg\xi(\mathfrak{B}_3) \wedge \neg\xi(\mathfrak{B}_4) \wedge \neg\xi(\mathfrak{B}_5)$$

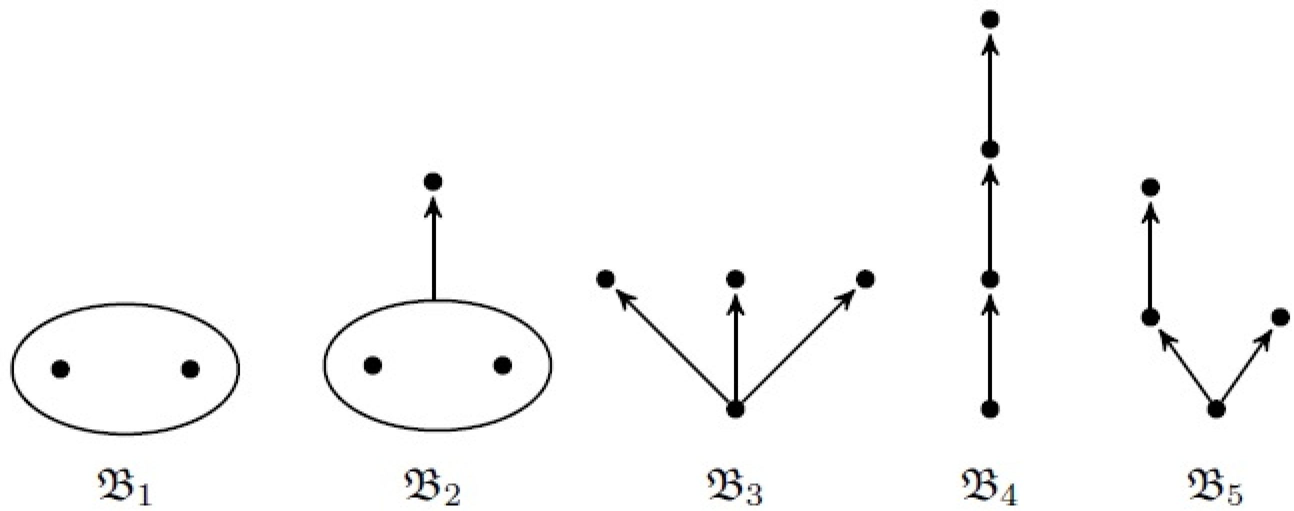
Axiomatization



Lemma 1: Each crown frame validates the axiom ξ.
Lemma 2: Each rooted finite frame $\mathfrak{G}$ with $\mathfrak{G} \models \xi$ is a subreduction of some crown frame.

Theorem: The logic $\mathbf{PL_2}$ is axiomatized by the formula ξ.

Shorter Axioms



(I) $p \to \Box[\neg p \to \Box(p \to \Box p)]$

(II) $\Box[(r \wedge q) \to \gamma] \to [(r \wedge q) \to \Diamond(\neg(r \wedge q) \wedge \Box\Diamond p \wedge \Box\Diamond\neg p)]$

Where γ is the formula

$\Diamond\Box(p \wedge q) \wedge \Diamond\Box(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q)$.

Complexity

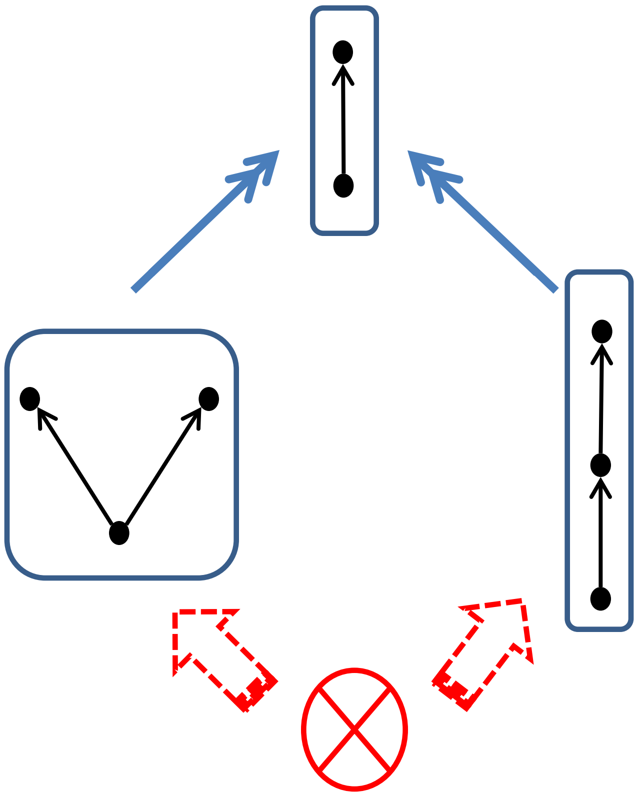
Theorem: The satisfiability problem of our logic is PSpace-complete.

Wolter, F. and M. Zakharyaschev, Spatial reasoning in RCC-8 with boolean region terms, in: Proc. ECAI, 2000, pp. 244-250

Craig Interpolation

(A) $\Box(r \rightarrow \Diamond(\neg r \wedge p \wedge \Diamond\neg p))$
(C) $(r \wedge \Diamond\Box\, s \wedge \Diamond\Box\, \neg s) \rightarrow \Diamond(\neg r \wedge \Diamond\Box\, s \wedge \Diamond\neg s)$

$A \rightarrow C$ is valid in $\mathbf{PL_2}$.



Further research

- Natural generalizations for spaces of higher dimension. $\mathbf{PL_n}$.
- Also d-logics and stronger languages.

Thank You