Modal logic of the planar polygons

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[1]

Abstract

We study the modal logic of the closure algebra P_2, generated by the set of all polygons of the Euclidean plane $\mathbb{R}^2$. We show that the logic is finitely axiomatizable, it is complete with respect to the class of all finite "crown" frames we define, it does not have the Craig interpolation property and its validity problem is PSPACE-complete.

Keywords: Modal Logic, topology, planar polygons.

Connections between modal logic and general topology were discovered and investigated in the works of McKinsey and Tarski in the 1940s (see, e.g., their seminal paper [9]). From the beginning it was established that if the modal diamond is interpreted as the closure operator over a topological space, then the minimal modal logic is **S4**. Moreover, the modal logic of any Euclidean space $\mathbf{R}^n$ is also **S4**. This result can be interpreted to mean that the modal language is not expressive enough to distinguish the Euclidean spaces from each other, in

[1] The authors wish to thank the members of Esakia Seminar for useful discussions on the topic. The work of the first author was supported by the Rustaveli Science Foundation grant # FR/489/5-105/11.

particular the modal language is insensitive to dimension. One can increase the expressive power by extending the language (e.g. by adding the global modality, or the difference operator) or change the interpretation of the modal diamond to another topological operator (for instance - the limit operator, as suggested already by McKinsey and Tarski in [9]). Another road to take is to restrict the *valuations*, allowing the propositional letters to denote not arbitrary subsets of the space in question, but some *well-behaved* subsets, where 'well-behaved' might carry a topological meaning like 'regular closed' or a more geometrical meaning like 'convex' or 'polygonal'. In [3,2] authors considered the collections of serial, convex, hyper-rectangular and chequered subsets of various Euclidean spaces and calculated the arising modal logics. In [10] the regular closed regions of (Euclidean) spaces were considered from the mereotopological point of view and in [6] the polygonal regions were considered again from the point of view of mereotopology (see also [5,7] for the recent developments in this direction). In the latter line of work however, the boolean operations on regions do not always coincide with the usual set-theoretic operations. In particular, the meet of two regular closed regions may not coincide with their intersection. To phrase it algebraically, the propositional part of the modal language is being interpreted in the boolean algebra of regular closed subsets, rather than in the powerset boolean algebra. This differs from our approach, where we work in the powerset algebra, as, for example, in [3].

In this paper, we consider pure modal language interpreted over the two-dimensional Euclidean plane $\mathbf{R}^2$ in such a way that the propositional letters denote only the so-called *polygonal* regions of the plane. The resulting modal logic is denoted by $\mathbf{PL_2}$. We prove the finite model property for $\mathbf{PL_2}$ and show that it is finitely axiomatizable and hence, decidable. The complexity of the satisfiability problem for $\mathbf{PL_2}$ is shown to be in PSPACE.

1 Preliminaries

Syntactically we are dealing with the basic modal language with the set of countably many propositional letters PROP and the formulas built in the usual way using the propositional connectives together with unary modal connectives $\Diamond$ and $\Box$.

Semantically our main object of study is the *polygonal plane* which we now proceed to define. Consider the regions of the plane obtained by the intersections of finitely many half-planes and generate the boolean algebra using the set-theoretic operations from these regions. An arbitrary member of the obtained boolean algebra is called a *polygon* and the collection of all polygons is denoted by P_2. The structure $\mathfrak{P}_2 = (\mathbb{R}^2, P_2)$ is called the *polygonal plane*.

A typical bounded member of P_2 is a finite union of (open) n-gons, line segments and points. In other words, we consider as entities not only the 2-dimensional n-gons, but also their boundaries, i.e. 'polygons' of lower dimension. P_2 turns out to be a closure subalgebra of the full closure algebra of all subsets of the real plane.

To interpret the modal language over the polygonal plane, we allow for valuations ν : PROP $\to P_2$ to range over polygons only. The valuations are extended to arbitrary modal formulas using the set-theoretic counterparts for the propositional connectives, interpreting $\Diamond$ as the topological closure, and $\Box$ as topological interior.

The set of all valid modal formulas over $\mathfrak{P}_2$ is denoted by $\mathbf{PL_2}$.

Our main object of study is the modal logic $\mathbf{PL_2}$.

2 Main findings

We examine what kind of finite frames does the logic $\mathbf{PL_2}$ have. To this end, we look at finite interior (open and continuous) images of the polygonal plane. Note that since we are talking about the images of the *polygonal* plane, the map must be such that the pre-image of each point is a polygon.

The typical examples are the following frames we call *crown frames.*

Definition 2.1 A *crown frame* $\mathfrak{S}_n$ is a frame (S_n, Q_n) such that $S_n = \{r, s_1, \cdots, s_{2n}\}$ and Q_n is the reflexive closure of the following:

$rQ_n s_i$	for all $s_i \in S_n$
$s_i Q_n s_j$	when $i < 2n$ is even and $j = i-1, i+1$
$s_{2n} Q_n s_1$	and $s_{2n} Q_n s_{2n-1}$

The general crown frame is depicted on figure 1.

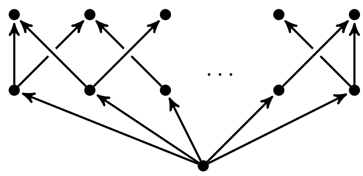

Fig. 1. Crown frame

We show that the logic of the crown frames coincides with $\mathbf{PL}_2$.

Theorem 2.2 *Any crown frame is an interior image of the polygonal plane.* [2]

Theorem 2.3 *Let ϕ be satisfiable on a polygonal plane. Then ϕ is satisfiable on one of the crown frames.*

Corollary 2.4 *The logic $\mathbf{PL}_2$ is determined by the class of finite crown frames. Hence this logic has fmp.*

Therefore, our task of investigating $\mathbf{PL}_2$ can be reduced to the setting of Kripke semantics by considering the logic of the crown frames.

To axiomatize $\mathbf{PL}_2$ we use Jankov-Fine formulas for finite rooted frames that are *not* $\mathbf{PL}_2$-frames. Firstly, we describe the five simplest frames that falsify $\mathbf{PL}_2$. These are depicted below.

[2] We view a crown frame as an Aleksandrov topological space obtained by declaring its upwards closed subsets open.

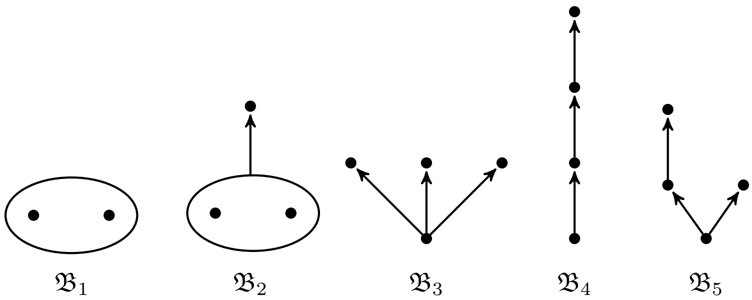

Fig. 2. Forbidden frames

We claim that the logic axiomatized by the Jankov-Fine axioms of these five frames coincides with $\mathbf{PL}_2$. Let ξ be the conjunction of negations of Jankov-Fine formulas for the Forbidden frames and let Ξ denote the extension of **S4** with ξ. Since Ξ is of finite depth, it is an easy consequence of Segerberg's theorem (see, e.g., [4, Theorem 8.85]) that:

Theorem 2.5 *The logic Ξ has the finite model property.*

Since both Ξ and $\mathbf{PL}_2$ are characterized by their finite rooted frames, to show the equality the following two lemmas suffice.

Lemma 2.6 *Each crown frame validates the axiom ξ.*

The above lemma demonstrates that $\Xi \subseteq \mathbf{PL}_2$. To address the other direction, recall that a frame $\mathfrak{F}$ is a *subreduction* of a frame $\mathfrak{G}$ if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

Lemma 2.7 *Each rooted finite frame $\mathfrak{G}$ with $\mathfrak{G} \models \xi$ is a subreduction of some crown frame.*

It follows from the two lemmas that:

Theorem 2.8 *The logic $\mathbf{PL}_2$ is axiomatized by the formula ξ above* **S4**. *In other words, $\mathbf{PL}_2 = \Xi$.*

We also present a slightly more intuitive and concise axiomatization of $\mathbf{PL}_2$ by the following two formulas:

$$\text{(I)}\ \ p \to \Box[\neg p \to \Box(p \to \Box p)]$$
$$\text{(II)}\ \ \Box[(r \wedge q) \to \gamma] \to [(r \wedge q) \to \Diamond(\neg(r \wedge q) \wedge \Diamond\Box p \wedge \Diamond\Box\neg p)]$$

Where γ is the formula

$$\Diamond\Box(p \wedge q) \wedge \Diamond\Box(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q).$$

The formula (I) forbids frames $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{B}_4$, while (II) forbids $\mathfrak{B}_3$ and $\mathfrak{B}_5$. We also show that all crown frames validate both (I) and (II), thereby proving:

Theorem 2.9 *The logic $\mathbf{PL}_2$ is axiomatized by* (I) *and* (II) *over* **S4**. *In other words, $\mathbf{PL}_2 = \mathbf{S4} + \text{(I)} + \text{(II)}$.*

Note that (I) carries an interesting dimensional meaning. Denote by $\delta A = \mathbb{C}A - A$ the *external boundary* of A (closure of A minus A). Then a space X validates (I) iff $\delta^3 A = \emptyset$ for all A. If A is a polygon, then δA is a polygon of strictly lower dimension. So over the polygonal plane $\delta^3 A = \emptyset$.

From the fmp and the finite axiomatization we conclude that our logic $\mathbf{PL}_2$ is decidable. Moreover, we calculate the computational complexity of the satisfiability problem.

Theorem 2.10 *The satisfiability problem of* $\mathbf{PL}_2$ *is* PSPACE*-complete.*

The PSPACE lower bound shown in [10] also extends to crown frames. The matching upper bound can be shown using an algorithm which in PTIME guesses a set of mosaics satisfying the formula, and which uses PSPACE to check whether each pair of mosaics can be connected.

Only very few extensions of **S4** have the Craig interpolation property [8] and $\mathbf{PL}_2$ is not among them, as the following counterexample shows. Consider the following two formulas:

$$(A)\ \Box(r \to \Diamond(\neg r \wedge p \wedge \Diamond\neg p))$$
$$(C)\ (r \wedge \Diamond\Box s \wedge \Diamond\Box\neg s) \to \Diamond(\neg r \wedge \Diamond\Box s \wedge \Diamond\Box\neg s)$$

$A \to C$ is valid in $\mathbf{PL}_2$. On any model making r and A true at the root, A forces a chain of depth 2. The consequent C then follows because of the nature of crown frames. There is no interpolant because the 2-element frame in which r is only true at the root is a p-morphic image preserving only (the common vocabulary) r of each non trivial model of A and of each model of $\neg C$.

Theorem 2.11 $\mathbf{PL}_2$ *does not have Craig interpolation property.*

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